2017
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,…, P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with scissors.
6. You may use the blank sheets of the problem booklet for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Find the value of the following definite integral:

$$I=\int_2^4 \frac{dx}{\sqrt{(x-2)(4-x)}}. \tag{1}$$

II. Find the general solution and the singular solution of the following differential equation:

$$y=x\frac{dy}{dx}+\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2. \tag{2}$$

III. Find the general solution of the following differential equation:

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}-8y=x^2. \tag{3}$$

Problem 2

Answer the following questions about the square matrix A of order 3:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -1 \\ -2 & -2 & 1 \end{pmatrix}. \qquad (1)$$

I. Find all eigenvalues of A.

II. Find the matrix A^n, where n is a natural number.

III. The square matrix $\boldsymbol{B}$ of order 3 is diagonalizable and meets $\boldsymbol{AB} = \boldsymbol{BA}$. Prove that any eigenvector $\boldsymbol{p}$ of $\boldsymbol{A}$ is also an eigenvector of $\boldsymbol{B}$.

IV. Find the square matrix $\boldsymbol{B}$ of order 3 that meets $\boldsymbol{B}^2 = \boldsymbol{A}$, where $\boldsymbol{B}$ is diagonalizable and all eigenvalues of $\boldsymbol{B}$ are positive.

V. The square matrix $\boldsymbol{X}$ of order 3 is diagonalizable and meets $\boldsymbol{AX} = \boldsymbol{XA}$. When $\mathrm{tr}(\boldsymbol{AX}) = d$, find the maximum of $\det(\boldsymbol{AX})$ as a function of d.

Here, d is positive real and all eigenvalues of $\boldsymbol{X}$ are positive. In addition, $\mathrm{tr}(\boldsymbol{M})$ is the trace (the sum of the main diagonal elements) of the square matrix $\boldsymbol{M}$, and $\det(\boldsymbol{M})$ is the determinant of the matrix $\boldsymbol{M}$.

Problem 3

Answer the following questions. Here, i, e, and $\log$ denote the imaginary unit, the base of the natural logarithm, and the natural logarithm, respectively.

I. Consider the definite integral I expressed as

$$I = \int_0^{2\pi} \frac{\cos\theta \, d\theta}{(2+\cos\theta)^2}. \tag{1}$$

1. Find a complex function $G(z)$ of a complex variable z when we rewrite I as an integral of a complex function as

$$\oint_{|z|=1} G(z) dz, \tag{2}$$

   where the integration path is a unit circle in the counter clockwise direction.
2. Find all poles and the respective orders and residues.
3. Evaluate the integral I.

II. Let a function of a real variable θ with real parameters α and β be

$$f(\theta;\alpha,\beta) = 1 + e^{2i\beta} + \alpha e^{i(\theta+\beta)}. \tag{3}$$

Consider the definite integral

$$F(\alpha,\beta) = \int_0^{2\pi} d\theta \frac{d}{d\theta}[\log f(\theta;\alpha,\beta)]. \tag{4}$$

1. Find a complex function $G(z)$ of a complex variable z when we rewrite $F(\alpha,\beta)$ as an integral of a complex function as

$$\oint_{|z|=1} G(z) dz, \tag{5}$$

   where the integration path is a unit circle in the counter clockwise direction.

2. Find all poles and the respective orders and residues.
3. Evaluate $F(\alpha, \beta)$ by classifying cases with respect to α and β. Ignore the case in which the integration path passes through any poles.

Problem 4

For the real numbers θ and α within the regions $0 \le \theta < 2\pi$ and $0 \le \alpha \le \pi$, consider the line L that passes through two points: point P($\cos\theta$, $\sin\theta$,1) and point Q($\cos(\theta+\alpha)$, $\sin(\theta+\alpha)$, -1) in a three-dimensional Cartesian coordinate system xyz.

I. Represent the line L as a linear function of a parameter t. Here, the point on the line L at $t=0$ should represent the point Q and the point at $t=1$ should represent the point P.

II. Find the surface S swept by the line L as an equation of x, y and z when θ varies in the region $0 \le \theta < 2\pi$. Let C be the intersection lines of the surface S with the plane $y=0$. Find the equation of C in terms of x and z, and sketch the shape of C.

Next, examine the Gaussian curvature of the surface S. Generally, when the position vector $\boldsymbol{r}$ of a point R on a curved surface is represented using parameters u and v by

$$\boldsymbol{r}(u,v) = (x(u,v), y(u,v), z(u,v)), \tag{1}$$

the Gaussian curvature K is represented as the following equation:

$$K = \frac{(\boldsymbol{r}_{uu}\cdot\boldsymbol{e})(\boldsymbol{r}_{vv}\cdot\boldsymbol{e}) - (\boldsymbol{r}_{uv}\cdot\boldsymbol{e})^2}{(\boldsymbol{r}_u\cdot\boldsymbol{r}_u)(\boldsymbol{r}_v\cdot\boldsymbol{r}_v) - (\boldsymbol{r}_u\cdot\boldsymbol{r}_v)^2}, \tag{2}$$

where $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are first-order partial differentials of $\boldsymbol{r}(u,v)$ with respect to the parameters u and v, and $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{uv}$ and $\boldsymbol{r}_{vv}$ are second-order partial differentials of $\boldsymbol{r}(u,v)$ with respect to the parameters u and v. $(\boldsymbol{a}\cdot\boldsymbol{b})$ represents the inner product of two three-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{e}$ is the unit vector of the normal direction at the point R.

III. Let the point W be the intersection of the surface S and the x axis in the region $x>0$. Calculate the Gaussian curvature of S at the point W for α within the region $0 \le \alpha < \pi$.

IV. For α within the region $0 \leq \alpha < \pi$, prove that the Gaussian curvature is less than or equal to 0 at arbitrary points on the surface S.

Problem 5

The Laplace transform $F(s) = L[f(t)]$ of a function $f(t)$, where $t \geq 0$, is defined as

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt . \tag{1}$$

Here, s is a complex number, and e is the base of the natural logarithm. Answer the following questions. Show the derivation process with your answer.

I. Prove the following relations:
   1. $L\left[t^n\right] = \dfrac{n!}{s^{n+1}}$, where n is a natural number.
   2. $L\left[\dfrac{df(t)}{dt}\right] = sF(s) - f(0)$, where $f(t)$ is a differentiable function.
   3. $L[e^{at} f(t)] = F(s-a)$, where a is a real number.

II. Solve the following differential equation using a Laplace transformation for $t \geq 0$:

$$t\frac{d^2 f(t)}{dt^2} + (1+3t)\frac{df(t)}{dt} + 3f(t) = 0,\quad f(0) = 1,\quad \left.\frac{df}{dt}\right|_{t=0} = -3 . \tag{2}$$

You can use the relation $L[tf(t)] = -\dfrac{d}{ds}F(s)$, if necessary.

III. The point $\mathrm{P}(x(t), y(t))$, which satisfies the following simultaneous differential equations, passes through the point (a,b) when $t = 0$. a and b are real numbers.

$$\begin{cases} \dfrac{dx(t)}{dt} = -x(t) \\ \dfrac{dy(t)}{dt} = x(t) - 2y(t) . \end{cases} \tag{3}$$

   1. Solve Equation (3) using a Laplace transformation for $t \geq 0$.

2. Express the relation between x and y by eliminating t from the solution of III. 1.
3. For both $(a,b) = (1,1)$ and $(-1,1)$, draw the trajectories of point P when t varies continuously from 0 to infinity.

Problem 6

A product factory manufactures 2 types of products: *product-I* and *product-II*. *Part-A* is necessary for *product-I*, and both *part-A* and *part-B* are necessary for *product-II*. There are parts that have standard quality and parts that do not have standard quality among *part-A* and *part-B*. All parts are delivered from the part factory to the product factory, but there is no quality check of any part. The qualities of *part-A* and *part-B* are independent, and they will not affect each other. The probabilities that *part-A* and *part-B* have standard quality are a and b, respectively.

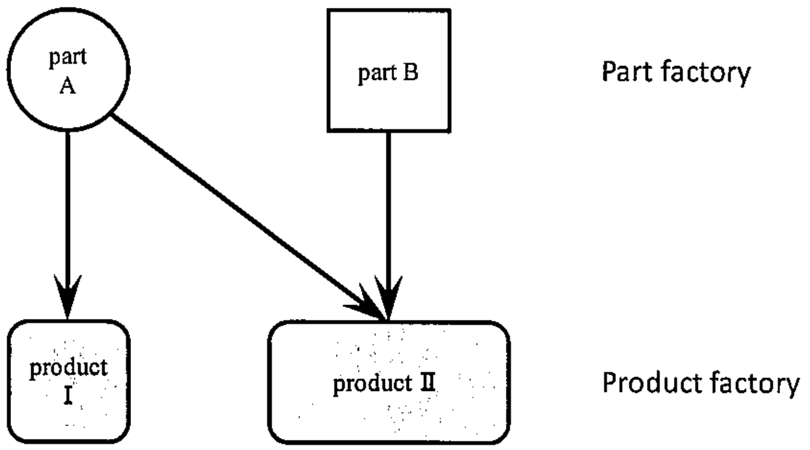

Figure 6.1

A final quality inspection is made in the product factory for *product-I* and for *product-II* before shipment. The inspection judges whether the quality of each product meets the standard or not. The inspections will not affect each other. The product inspection is not perfect: namely, products that have standard quality pass the product inspection as acceptable with the probability x. The products that do not have standard quality pass the product inspection as acceptable with the probability y.

Answer the following questions:

I. A *product-I* is randomly sampled and inspected once. Here, the probability that *product-I* can be manufactured with standard quality is defined as follows:

- The probability that *product-I* has standard quality is c if *part-A* has standard quality.
- *Product-I* will never have standard quality if *part-A* does not have standard quality.

1. Show the probability that the selected *product-I* passes the product inspection as acceptable.

2. Show the probability that the selected *product-I* actually has standard quality after it has passed the product inspection as acceptable.

II. A *product-II* is randomly sampled and inspected n times. Here, the probability that *product-II* can be manufactured with standard quality is defined as follows:

- The probability that *product-II* has standard quality is c if both *part-A* and *part-B* have standard quality.
- The probability that *product-II* has standard quality is d if only either *part-A* or *part-B* has standard quality.
- *Product-II* will never have standard quality if both *part-A* and *part-B* do not have standard quality.

1. Show the probability that the selected *product-II* has standard quality.
2. Show the probability that the selected *product-II* actually has standard quality after it has passed all product inspections (i. e., n times) as acceptable.