2012
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets whether in English or Japanese until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer three problems out of the six problems in the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,…, P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Obtain the general solution of the following differential equation:

$$\frac{dy}{dx} + \left(2x^2 + 1\right)y + y^2 + \left(x^4 + x^2 + 2x\right) = 0. \qquad (1)$$

You may use the particular solution given by $y = -x^2$.

II. Let $y(x)$ be a real function continuous up to second-order differential and $y' = dy/dx$. Find $y(x)$, which gives the extremum of a functional

$$I(y) = \int_{x_1}^{x_2} F(x, y, y')dx . \qquad (2)$$

Answer the following questions.

1. Define $Y(x)$ as $Y(x) = y(x) + k\eta(x)$, where $\eta(x)$ is an arbitrary differentiable function, and k is a small real number ($|k| << 1$). Obtain $\delta I = I(Y) - I(y)$ in terms of expansion of k up to the first order.
2. Assume at the ends of integration of Eq. (2), $y(x_1) = y_1$ and $y(x_2) = y_2$. Based on the result of Question II.1, express the condition which gives the extremum of $I(y)$ by using F, x, y and y'.
3. Assume $y(x_1) = y_1$ is given only at one end of integration of Eq. (2). Obtain the condition which gives the extremum of $I(y)$.
4. When $F(x, y, y') = y'^2 + y^2$, $x_1 = 0$, $x_2 = 1$ and $y(x_1) = 1$, obtain the function $y(x)$ which gives the extremum of $I(y)$.

Problem 2

Let $\mathbf{A}$ be the following matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & \alpha \end{pmatrix},$$

where α is a real number greater than or equal to zero $(\alpha \geq 0)$. Answer the following questions.

I. When $\alpha = \dfrac{5}{2}$, obtain all of the eigenvalues of the matrix $\mathbf{A}$ and corresponding eigenvectors whose norm sizes are equal to 1.

II. When α is a positive real number $(\alpha > 0)$, let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix $\mathbf{A}$.

1. Express λ_1, λ_2 and λ_3 by using α and show that all of them are real numbers.
2. Obtain $\lim\limits_{\alpha \to \infty} \dfrac{\max(\lambda_1, \lambda_2, \lambda_3)}{\alpha}$, where $\max(\lambda_1, \lambda_2, \lambda_3)$ is the maximum of λ_1, λ_2 and λ_3.
3. Obtain $\lim\limits_{\alpha \to \infty} \min(\lambda_1, \lambda_2, \lambda_3)$, where $\min(\lambda_1, \lambda_2, \lambda_3)$ is the minimum of λ_1, λ_2 and λ_3.

III. When $\alpha = 0$, obtain matrix elements of $\mathbf{A}^n$. Here n is a positive integer.

Problem 3

Answer the following questions concerning complex functions. Let i be the imaginary unit and e the base of natural logarithm.

I. Consider the mapping from the z plane $(z = x + iy)$ to the ζ plane $(\zeta = \xi + i\eta)$ using the complex function given by

$$f(z) = \frac{1+z}{1-z} \quad . \tag{1}$$

Prove that the function $\zeta = f(z)$ transforms the region inside a unit circle on the z plane, $|z| < 1$, to the right half of the ζ plane, $\mathrm{Re}\,\zeta > 0$. Also find the region on the ζ plane, to which a point on the circumference of the unit circle on the z plane, $z = e^{i\varphi}$ $(0 < \varphi < 2\pi)$, is transformed by $\zeta = f(z)$.

II. A function $h(\zeta)$ is regular where $\mathrm{Re}\,\zeta > 0$ and is also continuous where $\mathrm{Re}\,\zeta \geq 0$ on the ζ plane. Let $\zeta_0 = \xi_0 + i\eta_0$ and $\zeta_1 = -\xi_0 + i\eta_0$, where $\xi_0 > 0$ and $-\infty < \eta_0 < +\infty$. A closed path of integral C is shown in Figure 3.1, which consists of a half circle C_R with radius R $(R^2 > \xi_0^2 + \eta_0^2)$ and the diameter C_i of the half circle. The direction of the integral path is negative (clockwise). Equation (2) holds in this case.

$$\begin{cases} h(\zeta_0) = -\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{h(\zeta)}{\zeta - \zeta_0} d\zeta \\ 0 = -\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{h(\zeta)}{\zeta - \zeta_1} d\zeta \end{cases} \tag{2}$$

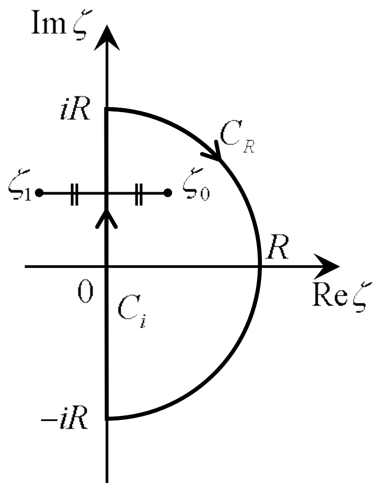

Figure 3.1

By using Eq. (2), prove Eq. (3) when $h(\zeta)$ satisfies $\lim_{|\zeta|\to\infty} |h(\zeta)| = 0$. Show the derivation process also.

$$h(\xi_0 + i\eta_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\xi_0}{\xi_0^2 + (\eta_0 - t)^2} h(it)dt \tag{3}$$

III. A function $k(z)$ is regular where $|z|<1$ and is also continuous where $|z|\le 1$ on the z plane. Prove that Eq. (4) holds for arbitrary values of r and θ which satisfy $0 \le r < 1$ and $0 < \theta < 2\pi$, respectively. Show the derivation process also.

$$k(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} k(e^{i\varphi})\,d\varphi \tag{4}$$

Problem 4

Let regions A, B and C be defined by Eqs. (1), (2) and (3) in the rectangular Cartesian coordinate system xyz, respectively.

$$x^2 + y^2 \le r^2, \tag{1}$$
$$y^2 + z^2 \le r^2, \tag{2}$$
$$z^2 + x^2 \le r^2, \tag{3}$$

where $r > 0$.

Answer the following questions.

I. Region D is defined as an intersection of the regions A and B.
   1. A plane $y = t$ $(0 \le t \le r)$ is cutting the region D. Obtain the cross sectional area.
   2. Obtain the volume and the surface area of the region D.

II. Region E is defined as an intersection of the regions A, B and C.
   1. Obtain the maximum value of $x^2 + y^2 + z^2$ within the region E. Moreover, list all of the points that give the maximum value.
   2. Obtain the volume and the surface area of the region E.

Problem 5

Suppose $F(\omega)$ is the Fourier transform of $f(x)$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-i\omega x) dx \, . \tag{1}$$

The inverse Fourier transform of $F(\omega)$ is given as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega x) d\omega \, . \tag{2}$$

Here, i is the imaginary unit.

Answer the following questions.

I. Calculate $F(\omega)$ for the following three Eqs. (4), (5) and (6).
Here, $g(x)$ is given as

$$g(x) = \begin{cases} 1 - \dfrac{|x|}{d} & |x| \le d \\ 0 & |x| > d \end{cases} \, . \tag{3}$$

The constants a and d satisfy $0 < d < \frac{a}{2}$, and N is a positive integer.

1. $f(x) = g(x)$. (4)
2. $f(x) = g(x-a) + g(x) + g(x+a)$. (5)
3. $f(x) = \sum_{n=-N}^{N} g(x - na)$. (6)

II. Calculate $\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$ when $f(x)$ is given by Eq. (6).

Problem 6

I. A particle is moving along the number line. The particle starts from the origin 0, and then at each timing step, it moves randomly by -1, 0 or $+1$ with equal probability of $1/3$. Answer the following questions.

1. Obtain the mean and variance of the position of the particle after n timing steps, where n is a positive integer.
2. Let L be a positive integer. The particle stops moving when it reaches $-L$ or $+L$ on the number line. When the particle is located at k, the expected number of steps required to stop moving is expressed by $e(k)$, where k is an integer satisfying $-L < k < L$. Derive the difference equation in terms of $e(k)$, $e(k-1)$ and $e(k+1)$.
3. The solution for the difference equation derived in Question I.2 can be expressed by a quadratic function of k. Solve the equation and obtain $e(k)$. Use the terminal condition $e(-L) = e(L) = 0$.

II. A complete graph is defined as a graph in which every pair of distinct vertices is connected by an edge. A complete graph of *n* nodes is expressed as K_n. Figures 6.2 and 6.3 show example graphs of K_5 and K_6, respectively. Consider labeling a complete graph with two different labels. Figure 6.1 gives an example of labeling a complete graph K_4, in which different-labeled edges are represented as dotted lines and solid lines. Answer the following questions.

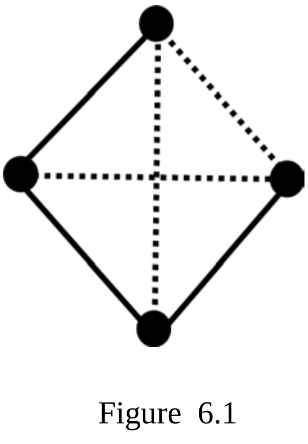

Figure 6.1

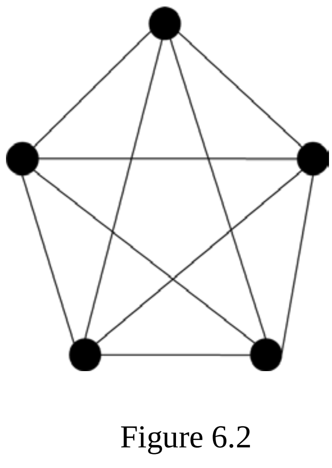

Figure 6.2

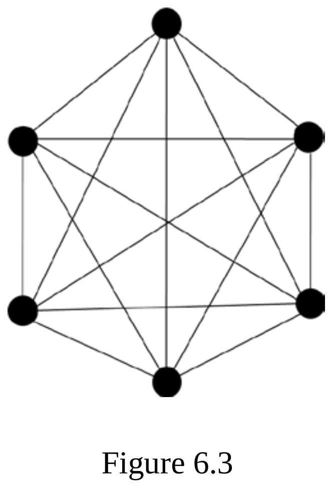

Figure 6.3

1. The graph labeling shown in Figure 6.1 does not include a complete graph K_3 with the same labeled edges. Label a complete graph K_5 shown in Figure 6.2 with two different labels such that it does not include a complete graph K_3 with the same labeled edges.

2. It is known that at a party of six people there are either three mutual acquaintances or three mutual strangers. Prove this claim by means of labeling a complete graph K_6 shown in Figure 6.3. Assume there is no one-sided acquaintance in the relationship.

3. Consider labeling a complete graph K_8 with two distinct labels. Then, provide an example of labeling this graph which does not include any of the following two complete graphs:
   - A complete graph K_3 with solid edges
   - A complete graph K_4 with dotted edges.