2015
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer three problems out of the six problems in the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Consider the differential equation,

$$\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - 2y = 0. \tag{1}$$

Using $p = \frac{dy}{dx}$, obtain the general and singular solutions of Eq. (1).

II. Consider the normal of the curve $y = y(x)$ at an arbitrary point $P(x, y(x))$. Let the point N be the intersection of the normal with the x-axis (see Fig. 1.1). Assume that the curvature radius R of the curve at the point P is the double of the length PN. The curvature radius R at the point P is given by Eq. (2):

$$R = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \left|\frac{d^2y}{dx^2}\right|. \tag{2}$$

Answer the following questions.

1. Show that $y(x)$ satisfies Eq. (3):

$$1 + \left(\frac{dy}{dx}\right)^2 = 2\left|y\frac{d^2y}{dx^2}\right|. \tag{3}$$

2. Solve Eq. (3) for $y\frac{d^2y}{dx^2} > 0$. Give the name of the kind of curve that the obtained equation represents.

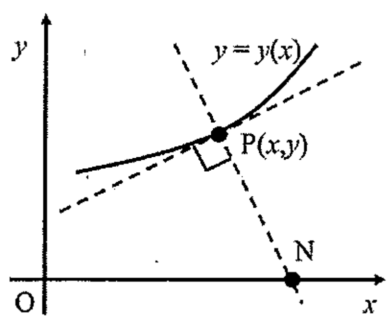

Fig.1.1

Problem 2

A quadratic form f of n real variables $x_i\ (i=1,2,\cdots,n)$ can be expressed as

$$f = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \tag{1}$$

where A is an $n \times n$ symmetric matrix, and $\boldsymbol{x}^{\mathrm{T}}$ represents the transpose of vector $\boldsymbol{x}$, i.e., a row vector consisting of $x_i\ (i=1,2,\cdots,n)$. The norm $\|\boldsymbol{x}\|$ of vector $\boldsymbol{x}$ is defined as $\|\boldsymbol{x}\| \equiv \sqrt{\sum_{i=1}^{n} x_i^2}$. Answer the following questions.

I. Obtain the 3×3 symmetric matrix A when the quadratic form of the following Eq. (2) is expressed with Eq. (1):

$$f = 6x_1^2 + 6x_2^2 + 5x_3^2 - 4x_2x_3 + 4x_3x_1 - 2x_1x_2 . \tag{2}$$

II. Consider the general $n \times n$ symmetric matrix A. Let $\lambda_i\ (i=1,2,\cdots,n)$ be eigenvalues of matrix A, and $\boldsymbol{u}_i\ (i=1,2,\cdots,n)$ be the corresponding eigenvectors of matrix A. When $\lambda_j \neq \lambda_k$, prove that $\boldsymbol{u}_j$ and $\boldsymbol{u}_k$ are orthogonal.

III. Consider an $n \times n$ square matrix $U = (\boldsymbol{u}_1 \boldsymbol{u}_2 \cdots \boldsymbol{u}_n)$ that consists of n column eigenvectors $\boldsymbol{u}_i\ (i=1,2,\cdots,n)$ of an $n \times n$ symmetric matrix A. Here, we assume that all $\boldsymbol{u}_i$ are mutually orthogonal and $\|\boldsymbol{u}_i\| = 1$ for all $\boldsymbol{u}_i$. Prove that $\|\boldsymbol{x}\| = \|U\boldsymbol{x}\|$ is satisfied for any vector $\boldsymbol{x}$.

IV. We define $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ as $\boldsymbol{y} = U\boldsymbol{x}$ by using U in III. Express the quadratic form f of Eq. (1) using $y_i\ (i=1,2,\cdots,n)$ and $\lambda_i\ (i=1,2,\cdots,n)$ in II. Show also the deriving process.

V. Under the condition of $\|\boldsymbol{x}\| = 1$, obtain the maximal value of the quadratic form f of Eq. (2) and find the corresponding $\boldsymbol{x}$.

Problem 3

For an arbitrary function $g(z)$, let us define the functions $g^{+}(x)$ and $g^{-}(x)$, respectively, as $g^{+}(x) \equiv \lim_{y \to +0} g(x+iy)$ and $g^{-}(x) \equiv \lim_{y \to -0} g(x+iy)$. Here, i is the imaginary unit, x and y are real numbers, and z is a complex number. For a continuous function $\varphi(x)$ in the interval $-1 \le x \le 1$, $f(z)$ is defined as:

$$f(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\varphi(x)}{x - z} dx . \qquad (1)$$

Answer the following questions.

I. For the real interval $-1 < x < 1$, show that Eq. (2) is satisfied:

$$f^{+}(x) - f^{-}(x) = \varphi(x) . \qquad (2)$$

II. Prove that the function $X(z) = \sqrt{z^2 - 1}$ satisfies the following relation on the real axis:

$$\frac{X^{+}(x)}{X^{-}(x)} = \begin{cases} -1 & (|x| < 1) \\ 1 & (|x| > 1) \end{cases} . \qquad (3)$$

Here, the branch cut of $X(z)$ is chosen along the real interval $-1 \le x \le 1$.

III. Derive $f(z)$ of Eq. (1) in the case of $\varphi(x) = X^{+}(x)$ with $X(z)$ defined in II.

Problem 4

Consider the three points, $P(\cos\theta, \sin\theta, 1)$, $Q(-\cos\theta, -\sin\theta, -1)$ and $R(\cos 2\theta, \sin 2\theta, -1)$, expressed in terms of a real number θ in the xyz orthogonal coordinate system with the origin O. Answer the following questions.

I. Calculate the length of the line segment $\overline{PQ}$.

II. Express the area S of the triangle PQR using θ.

III. Let the point M be the intersection point of the line segment $\overline{PR}$ with the xy plane. Express the coordinates of the point M using θ.

IV. Answer the following questions, supposing that θ varies continuously from 0 to π.

1. Sketch the locus of the point M on the xy plane.
2. Calculate the area of the region swept by the line segment $\overline{OM}$.

Problem 5

The Fourier transform $F(\omega)$ of a function $f(t)$ is defined as

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt, \tag{1}$$

where i is the imaginary unit, and t and ω are real numbers. Answer the following questions.

I. Prove that $F(-\omega) = \overline{F(\omega)}$ for any real function $f(t)$. Here, $\overline{F(\omega)}$ denotes the complex conjugate of $F(\omega)$.

II. The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(s) g(t-s) ds. \tag{2}$$

Here, s is a real number. Let $G(\omega)$ be the Fourier transform of $g(t)$. Express the Fourier transform of $(f * g)(t)$ using $F(\omega)$ and $G(\omega)$. Show also the deriving process.

III. Let us consider a real function $h(t)$ satisfying $h(t) = 0$ for $t < 0$ and its Fourier transform $H(\omega)$. Now we define an even function $r(t)$ and an odd function $x(t)$ as follows:

$$r(t) = \frac{h(t) + h(-t)}{2}, \tag{3}$$

$$x(t) = \frac{h(t) - h(-t)}{2}. \tag{4}$$

Using these two, $h(t)$ is expressed as $h(t) = r(t) + x(t)$. We use $R(\omega)$ and $X(\omega)$ to denote the Fourier transforms of $r(t)$ and $x(t)$, respectively. Answer the following questions.

1. Show that $R(\omega)$ is real for any ω.
2. Show that $X(\omega)$ is purely imaginary for any ω.
3. Let $h(t) = \begin{cases} \exp(-t) & (t \geq 0) \\ 0 & (t < 0) \end{cases}$ and calculate $R(\omega)$ and $X(\omega)$. Then, schematically draw each graph of the real part of $R(\omega)$ and the imaginary part of $X(\omega)$ as a function of ω.
4. Let $r(t) = \begin{cases} 1/4 & (|t| \leq 2) \\ 0 & (|t| > 2) \end{cases}$ and calculate $R(\omega)$ and $X(\omega)$. Then, schematically draw each graph of the real part of $R(\omega)$ and the imaginary part of $X(\omega)$ as a function of ω.

Problem 6

Consider the arrival time distributions of people coming to a restaurant. Each person comes one by one. Suppose that the n_0-th person arrives at the restaurant at the time t_0. Let $f_n(t)$ be the probability density of (n_0+n)-th person $(n \geq 1)$ arriving at the restaurant at the time (t_0+t). Here, $t>0$. Assume that $f_1(t)$ is given by

$$f_1(t) = \lambda e^{-\lambda t}, \qquad (1)$$

regardless of n_0 and t_0. Here, e is the base of the natural logarithm and λ is a positive constant. Answer the following questions.

I. Find the expectation value of the arrival time of the (n_0+1)-th person.

II. Show that $f_2(t) = \lambda^2 t e^{-\lambda t}$, $f_3(t) = \frac{1}{2}\lambda^3 t^2 e^{-\lambda t}$, and $f_4(t) = \frac{1}{6}\lambda^4 t^3 e^{-\lambda t}$.

III. Speculate the functional form of $f_n(t)$ from the results of II and prove it by mathematical induction.

IV. Let m be the number of the new persons coming to the restaurant from t_0 to t_0+T $(T>0)$, i.e., in the time interval (t_0, t_0+T). The probability distribution of m, $h(m,T)$, obeys the Poisson distribution:

$$h(m,T) = \frac{1}{m!}(\lambda T)^m e^{-\lambda T}. \qquad (2)$$

Calculate the expectation value of m. Here, the formula $e^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}$ may be used.