2014
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer three problems out of the six problems in the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Answer the following questions concerning the differential equation

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = R(t). \tag{1}$$

1. Obtain the general solution for $R(t) = 0$.
2. Obtain the general solution for $R(t) = 3t^2$.

II. Answer the following questions concerning the simultaneous differential equations below. Here, $\dot{x} = \dfrac{dx}{dt}$.

$$\begin{cases} \dfrac{d^2x}{dt^2} &= -2\dfrac{dx}{dt} - 3y + 2, \\ \dfrac{dy}{dt} &= \dfrac{dx}{dt} + 2y, \\ x(0) = 0, \ \dot{x}(0) = 0, \ y(0) = 2. \end{cases} \tag{2}$$

1. Let us transform the simultaneous differential equations in Eq.(2) into the form $\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, $\boldsymbol{x}(0) = \boldsymbol{c}$. Here, $\boldsymbol{x}$ is defined as

$$\boldsymbol{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \\ y(t) \end{pmatrix}. \tag{3}$$

   $\boldsymbol{A}$ is a constant matrix, and $\boldsymbol{b}$ and $\boldsymbol{c}$ are constant vectors. Obtain $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

2. Calculate all the matrix elements of $e^{t\boldsymbol{A}}$. Here, e denotes the base of the natural logarithm.
3. Using the results of Questions 1 and 2, solve the simultaneous differential equations in Eq.(2).

Problem 2

Let m and n be positive integers, where $m > n$. Consider a real matrix $\boldsymbol{A}$ with m rows and n columns. Answer the following questions, assuming that the rank of $\boldsymbol{A}$ equals n.

I. Prove that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is a symmetric matrix, where $\boldsymbol{A}^{\mathrm{T}}$ represents the transpose of matrix $\boldsymbol{A}$.

II. A real symmetric matrix $\boldsymbol{C}$ is said to be positive definite if $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x} > 0$ for any real column vector $\boldsymbol{x} \neq \boldsymbol{0}$. Prove that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is a positive definite matrix.

III. Show that all eigenvalues of any positive definite matrix are positive. With this result, prove that any positive definite matrix has an inverse matrix.

IV. Let $\boldsymbol{b}$ be an m-dimensional real column vector. A solution $\boldsymbol{x}$ that satisfies the linear equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ may not always exist. Therefore, as an approximate solution we consider $\boldsymbol{x}'$ which minimizes

$$\|\boldsymbol{A}\boldsymbol{x}' - \boldsymbol{b}\|^2 = (\boldsymbol{A}\boldsymbol{x}' - \boldsymbol{b})^{\mathrm{T}} (\boldsymbol{A}\boldsymbol{x}' - \boldsymbol{b}) . \tag{1}$$

Express such $\boldsymbol{x}'$ using $\boldsymbol{A}$ and $\boldsymbol{b}$.

Problem 3

I. Using the residue theorem, calculate the integral

$$\int_0^{\pi} \frac{\cos 4\theta}{1+\cos^2\theta}\, d\theta. \tag{1}$$

II. Calculate the integral

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z^2(1-z^2)}\, dz, \tag{2}$$

where e and i denote the base of the natural logarithm and the imaginary unit, respectively, and C is the closed path on the complex plane as shown in Fig. 3.1.

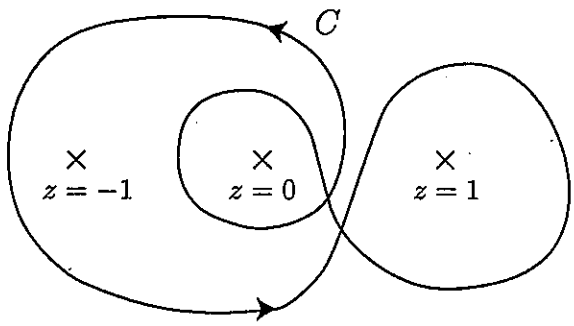

Fig. 3.1

III. Answer the following questions concerning the linear fractional transformation.

1. Find the respective domains onto which the two domains on the complex plane, $D_1 = \{z \mid |z| < 1\}$ and $D_2 = \{z \mid \mathrm{Re}\, z < 0\}$, are mapped under the linear fractional transformation $w = \dfrac{z+1}{z-1}$. Here, $\mathrm{Re}\, z$ denotes the real part of the complex number z.
2. Consider a linear fractional transformation $w = \dfrac{z-\alpha}{\alpha z - 1}$ such that the ring-like domain on the complex plane, $D_3 = \{z \mid \beta < |z| < 1\}$, is mapped onto the domain bounded by the two circles, $D_4 = \{w \mid |w - \frac{1}{4}| > \frac{1}{4},\ |w| < 1\}$. Obtain the values of α and β. Here, α and β are positive real numbers.

Problem 4

Answer the following questions concerning figures in xyz coordinates. Let $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ be unit vectors along the x-, y-, and z-axes, respectively.

I. Consider the line L_v that passes through the point $(0, 0, v)$ on the z-axis and has the direction $(\cos v, \sin v, 0)$. The parametric representation of L_v is given by the following equation using a parameter u:

$$\boldsymbol{L}_v(u) = u \cos v\, \boldsymbol{i} + u \sin v\, \boldsymbol{j} + v\, \boldsymbol{k}. \tag{1}$$

The locus of L_v for continuously varying v determines a surface S called ordinary helicoid. Using parameters u and v, the parametric representation of S is given by

$$\boldsymbol{S}(u, v) = u \cos v\, \boldsymbol{i} + u \sin v\, \boldsymbol{j} + v\, \boldsymbol{k}. \tag{2}$$

Find the normal vector of S at the point $\boldsymbol{S}(u, v)$.

II. Calculate the area of S in the region defined by $\{\boldsymbol{S}(u, v) | -1 \le u \le 1,\ 0 \le v \le 2\pi\}$ (cf. Fig. 4.1). If necessary, use a variable substitution $\{u \to t : u = \sinh t\}$ for the integration.

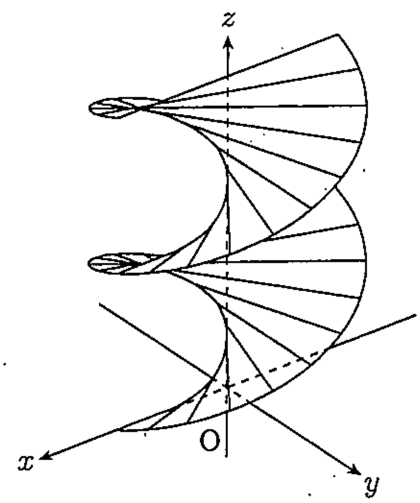

Fig. 4.1

III. By setting $u = 1$ in $\boldsymbol{S}(u, v)$, a curve R called ordinary helix is obtained with v as the parameter. R is represented by $\boldsymbol{R}(v) = \boldsymbol{S}(1, v) = \cos v\, \boldsymbol{i} + \sin v\, \boldsymbol{j} + v\, \boldsymbol{k}$. Consider the tangent T_v at the point $\boldsymbol{R}(v)$ on the curve R. Give the parametric representation $\boldsymbol{T}_v(w)$ of T_v with a parameter w, such that $\boldsymbol{T}_v(0) = \boldsymbol{R}(v)$.

IV. The locus of T_v for continuously varying v also determines a curved surface. The parametric representation of the surface D is given by the following equation using parameters v and w:

$$\boldsymbol{D}(v, w) = \boldsymbol{T}_v(w). \qquad (3)$$

1. Find the normal vector of D at the point $\boldsymbol{D}(v, w)$. Here, $w \neq 0$.
2. Show, for arbitrary $w_1 \neq 0$, $w_2 \neq 0$ and v, that the normal vectors of D at the points $\boldsymbol{D}(v, w_1)$ and $\boldsymbol{D}(v, w_2)$ are parallel to each other.

Problem 5

Answer the following questions. Show the derivation process with your answer. Note that e denotes the base of the natural logarithm.

I. Let t be a real number. The convolutional integral of two functions $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau. \tag{1}$$

Following the definition above, calculate the convolutional integrals of $f(t)$ and $g(t)$ given below. Here, ω is a real constant.

1. $f(t) = \cos(\omega t),\ g(t) = \sin(\omega t)$.
2. $f(t) = e^t,\ g(t) = e^{-t}$.

II. The Laplace transform $F(s) = L[f(t)]$ of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt, \tag{2}$$

where s is a complex number. Also, an equation

$$L[(f * g)(t)] = L[f(t)]\, L[g(t)] \tag{3}$$

holds for the convolutional integral $(f*g)(t)$ defined in Eq.(1). Now, we define two functions $q(t)$ and $r(t)$ as follows:

$$q(t) = u(t) - u(t-1), \tag{4}$$

$$r(t) = t + \sin(2\pi t), \tag{5}$$

where $u(t)$ is defined as

$$u(t) = \begin{cases} 0 & (t \le 0) \\ 1 & (t > 0) \end{cases}. \tag{6}$$

1. Draw in separate sketches $q(t)$ and $r(t)$ for the range $0 \le t < 4$.
2. Calculate the Laplace transforms of $q(t)$ and $r(t)$.

3. Let us define $y(t) = (q * r)(t)$. Calculate $y(t)$. Also sketch $y(t)$ for the range $0 \le t < 4$. If necessary, use the equality $L[f(t-a)u(t-a)] = e^{-as}F(s)$ for $a \ge 0$.

III. Find the solution $x(t)$ to the integral equation

$$x(t) + 2\,e^t \int_0^t x(\tau)\,e^{-\tau} d\tau = t\,e^t. \qquad (7)$$

Problem 6

Consider the probability $P_n(k)$ to have just k "heads" out of n tosses of a coin. The probabilities of "head" and "tail" for one toss are assumed to be p and $1-p$, respectively. Here, $0 < p < 1$ and $0 \le k \le n$, and e is the base of the natural logarithm. Answer the following questions.

I. Find $P_n(k)$.

II. The expected number of "heads" out of n tosses, μ, is defined as

$$\mu = \sum_{k=0}^{n} k P_n(k). \tag{1}$$

Prove that $\mu = np$.

III. Let x be a real number which fulfills $0 < x < 1$. Calculate

$$I(x) = \lim_{n\to\infty} \left[-\frac{1}{n} \log_e P_n(\lfloor xn \rfloor) \right]. \tag{2}$$

Here, $\lfloor y \rfloor$ stands for the largest integer not greater than the real number y. Also, find x at which $I(x)$ attains its minimum. Note that you may use Stirling's formula, $m! \approx \sqrt{2\pi m}(m/e)^m$ for integer $m \gg 1$.

IV. Let us define $\psi_n(\theta) = \log_e \left[\sum_{k=0}^{n} e^{\theta k} P_n(k) \right]$. Here, θ is a real number. Calculate $\psi_n(\theta)$. Also, let $\psi(\theta) \doteq \lim_{n\to\infty} \frac{1}{n}\psi_n(\theta)$ and calculate $\psi^*(\eta) = \max_\theta [\theta\eta - \psi(\theta)]$. Here, $0 < \eta < 1$, and $\max_\theta f(\theta)$ represents the maximum value of a function $f(\theta)$.