2020
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in English or Japanese.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of each answer sheet represent the problem number that you answer (P 1, P 2, …, P 6) and also the class of the course (master M, doctor D) that you are applying. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet.
6. You may use the blank sheets of the problem booklets for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklets or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Answer the following questions about the differential equation:

$$\cos x \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} - \frac{y}{\cos x} = 0 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right). \tag{1}$$

1. A particular solution of Eq. (1) is of the form of $y = (\cos x)^m$ (m is a constant). Find the constant m.
2. Find the general solution of Eq. (1), using the solution of Question I.1.

II. Find the value of the following integral:

$$I = \int_1^\infty x^5 e^{-x^4+2x^2-1} dx. \tag{2}$$

Note that, for a positive constant α, the relation $\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$ holds.

III. Express the general solution of the following differential equation in the form of $f(x, y) = C$ (C is a constant) using an appropriate function $f(x, y)$:

$$(x^3 y^n + x)\frac{dy}{dx} + 2y = 0 \quad (x > 0, y > 0), \tag{3}$$

where n is an arbitrary real constant.

Problem 2

Consider the following matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & \alpha \end{pmatrix}, \tag{1}$$

where α is a real number. In the following, the transpose of a vector $\boldsymbol{v}$ is denoted by $\boldsymbol{v}^{\mathrm{T}}$.

I. Obtain α when the sum of the three eigenvalues of the matrix $\boldsymbol{A}$ is 7.

II. Obtain α when the product of the three eigenvalues of the matrix $\boldsymbol{A}$ is -16.

III. Let $\|\boldsymbol{A}\|$ be the maximum of $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$ for the set of real vectors $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ that satisfy $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1$. Obtain α when $\|\boldsymbol{A}\| = 4$.

IV. In the following questions, $\alpha = 4$.

1. Obtain all eigenvalues of the matrix $\boldsymbol{A}$ and their corresponding normalized eigenvectors.
2. Find the range of $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{y}$ for the real vectors $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ that satisfy $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y} = 1$ and $y_1 - y_2 - 2y_3 = 0$.
3. Find the range of $\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}$ for the real vectors $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ that satisfy $\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z} = 1$ and $z_1 + z_2 + z_3 = 0$.

Problem 3

In the following, z denotes a complex number, and x and ε denote real numbers. The imaginary unit is denoted by i.

I. Answer the following questions about the function $f_n(z) = 1/(z^n - 1)$. Here, n is an integer greater than or equal to 2.

1. For the case that $n = 3$, find all singularities of $f_n(z)$.

2. Calculate the residue value at a singularity p_0 of $f_n(z)$ and give a simple expression of the residue in terms of n and p_0.

3. For a contour C given by the closed curve $|z| = 2$ and oriented in the counter-clockwise direction, evaluate the contour integral $\oint_C f_n(z)dz$.

II. Obtain the following limit value:

$$\lim_{\varepsilon\to+0}\left[\int_{-\infty}^{1-\varepsilon}\frac{1}{x^3-1}dx+\int_{1+\varepsilon}^{\infty}\frac{1}{x^3-1}dx\right]. \tag{1}$$

III. Obtain the following limit value:

$$\lim_{\varepsilon\to+0}\left[\int_{0}^{1-\varepsilon}\frac{\cos x}{x^4-1}dx+\int_{1+\varepsilon}^{\infty}\frac{\cos x}{x^4-1}dx\right]. \tag{2}$$

IV. Obtain the following limit value:

$$\lim_{\varepsilon\to+0}\left[\int_{0}^{1-\varepsilon}\frac{\sin\left(x^2-\frac{\pi}{4}\right)}{x^4-1}dx+\int_{1+\varepsilon}^{\infty}\frac{\sin\left(x^2-\frac{\pi}{4}\right)}{x^4-1}dx\right]. \tag{3}$$

Problem 4

In the three-dimensional orthogonal coordinate system xyz, the unit vectors along the x, y, and z directions are $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. Using the parameter θ $(0 \leq \theta \leq \pi)$, we define two curves by their vector functions $\mathbf{P}(\theta)$ and $\mathbf{Q}(\theta)$:

$$\mathbf{P}(\theta) = x(\theta)\mathbf{i} + y(\theta)\mathbf{j}, \tag{1}$$

$$\mathbf{Q}(\theta) = \mathbf{P}(\theta) + z(\theta)\mathbf{k}, \tag{2}$$

where

$$x(\theta) = \frac{3}{2}\cos(\theta) - \frac{1}{2}\cos(3\theta), \tag{3}$$

$$y(\theta) = \frac{3}{2}\sin(\theta) - \frac{1}{2}\sin(3\theta). \tag{4}$$

Here, $z(\theta)$ is a continuous function satisfying $z(0) > 0$ and $z(\pi) < 0$, and the curve parametrized by $\mathbf{Q}(\theta)$ lies on the sphere of radius 2, centered at the origin $(0, 0, 0)$ of the coordinate system. The positive direction of a curve corresponds to increasing values of the parameter θ. Note that the curvature is the reciprocal of the radius of curvature. Answer the following questions.

I. As θ is varied from 0 to π, calculate the arc length of the curve parametrized by $\mathbf{P}(\theta)$.

II. Obtain $z(\theta)$.

III. Let α be the angle between the tangent of the curve parametrized by $\mathbf{Q}(\theta)$ and the unit vector $\mathbf{k}$. Calculate $\cos(\alpha)$.

IV. Find the curvature $\kappa_P(\theta)$ of the curve parametrized by $\mathbf{P}(\theta)$. Here, $\theta = 0$ and $\theta = \pi$ are excluded.

V. Let $\kappa_Q(\theta)$ be the curvature of the curve parametrized by $\mathbf{Q}(\theta)$. Express $\kappa_Q(\theta)$ in terms of $\kappa_P(\theta)$ and α. Here, $\theta = 0$ and $\theta = \pi$ are excluded.

Problem 5

The Laplace transform of the function $f(t)$, defined for $t \geq 0$, is denoted by $F(s) = \mathcal{L}[f(t)]$ and its definition is given by

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) \exp(-st)\, dt, \tag{1}$$

where s is a complex number. In the following, the set of all complex numbers is denoted by $\mathbb{C}$, and the set of the complex numbers with positive real parts is denoted by $\mathbb{C}^+$.

I. Consider the following function $g(t)$ defined for $t \geq 0$:

$$g(t) = \int_0^{\infty} \frac{\sin^2(tx)}{x^2} dx. \tag{2}$$

1. Find the Laplace transform $G(s) = \mathcal{L}[g(t)]$ $(s \in \mathbb{C}^+)$ of the function $g(t)$.
2. Obtain the value of the following integral using the result of Question I.1:

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx. \tag{3}$$

II. Consider the function $u(x,t)$ that satisfies the following partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \quad (0 < x < 1, t > 0), \tag{4}$$

under the boundary conditions:

$$\begin{cases} \left.\dfrac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0 \quad (t \geq 0), & (5) \\ u(1,t) = 1 \quad (t \geq 0), & (6) \\ u(x,0) = \dfrac{\cosh(x)}{\cosh(1)} \quad (0 < x < 1). & (7) \end{cases}$$

1. The Laplace transform of $u(x,t)$ is denoted by $U(x,s) = \mathcal{L}[u(x,t)]$ ($s \in \mathbb{C}^+$). Derive the ordinary differential equation and boundary conditions for $U(x,s)$ with respect to the independent variable x. Here, the function $u(x,t)$ can be assumed to be bounded. The following relations can also be used:

$$\mathcal{L}\left[\frac{\partial u(x,t)}{\partial x}\right] = \frac{\partial U(x,s)}{\partial x}, \tag{8}$$

$$\mathcal{L}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \frac{\partial^2 U(x,s)}{\partial x^2}. \tag{9}$$

2. Using an analytic function $Q(s)$ ($s \in \mathbb{C}$), the function $U_c(x,s)$ is defined as follows:

$$U_c(x,s) = \frac{\cosh(x)}{(s-1)\cosh(1)} - \frac{\cosh(x\sqrt{s})}{Q(s)} \quad (0 \le x \le 1). \tag{10}$$

When the function $U(x,s) = U_c(x,s)$ satisfies the differential equation and the boundary conditions derived in Question II.1 for $s \in \mathbb{C}^+$, find the function $Q(s)$.

3. Using the function $Q(s)$ derived in Question II.2, the sequence of complex numbers $\{a_r\}$ ($r = 1, 2, \cdots$) is defined by arranging all of the roots of $Q(s) = 0$ ($s \in \mathbb{C}$) in ascending order of their absolute values. In this case, the following limits $R_r(x,t)$ are finite for $t \ge 0, 0 \le x \le 1$, and $r \ge 1$:

$$R_r(x,t) = \lim_{s \to a_r} (s - a_r) U_c(x,s) \exp(st), \tag{11}$$

and the solution of the partial differential equation (4) is given by

$$u(x,t) = \sum_{r=1}^{\infty} R_r(x,t). \tag{12}$$

Determine $R_1(x,t)$, $R_2(x,t)$, and $R_r(x,t)$ for $r \ge 3$.

Problem 6

Consider a game where points are awarded in n independent trials. In each trial, either $+1$ or -1 is awarded and both outcomes have the same probability of $1/2$. Let X_k be the point awarded in the k^{th} trial $(1 \le k \le n)$, and $S_k = \sum_{i=1}^{k} X_i$. In the following questions, n is an even integer such that $n \ge 4$, and t is an even integer such that $2 \le t \le n$.

I. Obtain the probability for $S_4 = 0$.

II. Let $P_n(t)$ be the probability for $S_n = t$. Find $P_n(t)$.

III. Let $P_n^+(t)$ be the probability for $S_1 = 1$ and $S_n = t$. Find $P_n^+(t)$.

IV. Let $P_n^-(t)$ be the probability for $S_1 = -1$ and $S_n = t$. Find $P_n^-(t)$.

V. Let $Q_n(t)$ be the probability that all of the variables $\{S_j\}$ $(j = 1, 2, \cdots, n-1)$ are greater than zero and $S_n = t$. Express $Q_n(t)$ with $P_n^+(t)$ and $P_n^-(t)$. Then, express $Q_n(t)$ with $P_n(t)$.

VI. Obtain the probability that all of the variables $\{S_j\}$ $(j = 1, 2, \cdots, n)$ are greater than zero.