2013
The Graduate School Entrance Examination
Mathematics
1:00 pm − 3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer three problems out of the six problems in the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Obtain the general solutions of the following differential equations:

1. $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 4e^{2x} \ . \tag{1}$$

2. $$x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 2x^4e^x \ . \tag{2}$$

Here e denotes the base of natural logarithm.

II. Calculate the value of the following integral:

$$\int_0^{\pi/2} \cos(2\theta)\ \log(\cos\theta)\ d\theta \ . \tag{3}$$

Problem 2

Answer the following questions concerning the real symmetric matrix $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} . \tag{1}$$

I. Obtain all of the eigenvalues and the corresponding eigenvectors of the matrix $\boldsymbol{A}$.

II. Obtain $\boldsymbol{A}^n$, where n is a natural number.

III. Let λ be a real number satisfying $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) \neq 0$, where $\boldsymbol{I}$ is a 3×3 unit matrix and $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ represents the determinant of $(\lambda \boldsymbol{I} - \boldsymbol{A})$. Let vector $\boldsymbol{x}$ be the solution to the linear equation

$$(\lambda \boldsymbol{I} - \boldsymbol{A})\, \boldsymbol{x} = \boldsymbol{b} \,, \tag{2}$$

where

$$\boldsymbol{b} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} . \tag{3}$$

Prove that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ can be written as

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \frac{3}{(\lambda-1)^2} + \frac{2}{(\lambda-2)^2} + \frac{6}{(\lambda-4)^2} \,, \tag{4}$$

where $\boldsymbol{x}^{\mathrm{T}}$ represents the transpose of $\boldsymbol{x}$.

Problem 3

Answer the following questions concerning complex functions. The imaginary unit is denoted by i.

I. Answer the following questions concerning the rational function defined on the complex z plane ($z = x + iy$) as

$$f(z) = \frac{1}{(z-1)z(z+2)} \ . \tag{1}$$

1. Obtain the Laurent expansion of $f(z)$ in the region on the complex z plane given by $1 < |z-1| < 3$.

2. Calculate the integral $\int_C f(z)dz$, where C is a closed integral path given by $|z-1| = 2$ and oriented in the counter-clockwise direction.

II. Answer the following questions concerning the rational function defined on the complex z plane ($z = x + iy$) as

$$g(z) = \frac{z^2}{z^4+1} \ . \tag{2}$$

1. Obtain all of the poles of $g(z)$ on the upper half-plane.

2. Calculate the definite integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx \tag{3}$$

by applying the residue theorem to $g(z)$.

Problem 4

A point is circulating with a constant period along an ellipse in a three-dimensional space. We consider the motion of the point observed from a rotating frame in the following procedure.

First, on a static orthogonal frame xyz, we define the motion of a point $P_0(x_0, y_0, z_0)$ along an ellipse C_0 on the xy-plane (Figure 4.1) using a parameter t as

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} a(-b+\cos t) \\ a\sqrt{1-b^2}\sin t \\ 0 \end{pmatrix}, \tag{1}$$

where the constants a and b fulfil $0 < a$ and $0 < b < 1$, respectively.

Next, we rotate the ellipse C_0 around the y-axis by an angle θ as shown in Figure 4.2. With this operation, the ellipse C_0 and the point P_0 are mapped to the ellipse C_1 and the point $P_1(x_1, y_1, z_1)$, respectively. Here, we assume $0 < \theta < \pi/2$ and

$$(1+b)\cos^2\theta \ < \ 1-b\,. \tag{2}$$

Finally, we consider a rotating orthogonal frame XYZ. As shown in Figure 4.3, the two frames share the origin, and the Z- and z-axes coincide. The angle between the X- and x-axes is equal to the parameter t used in Equation (1). The coordinate of the point P_1 on the frame XYZ is written as (X_1, Y_1, Z_1).

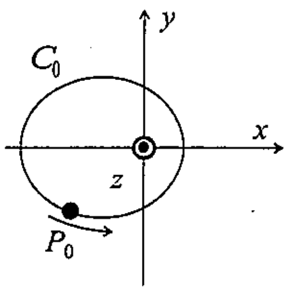

Figure 4.1

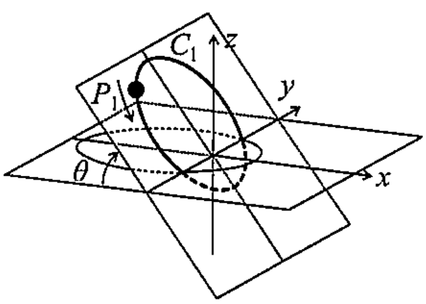

Figure 4.2

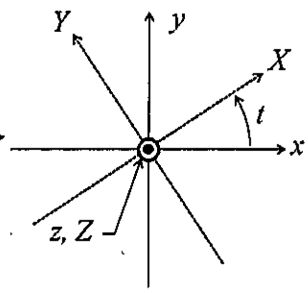

Figure 4.3

Answer the following questions.

I. Express the coordinate (x_1, y_1, z_1) of the point P_1 on the static frame xyz as functions of t.

II. Express the coordinate (X_1, Y_1, Z_1) of the point P_1 on the rotating frame XYZ as functions of t.

III. Obtain all the values of t (within the range $0 < t < 2\pi$) for which the point P_1 passes through the plane $Y = 0$. For each of the obtained values of t, find the corresponding coordinate of the point P_1 in the XYZ frame.

IV. Sketch the projection of the trajectory of P_1 on the XY plane, and mark the points corresponding to $t = \dfrac{k\pi}{4}$ $(k = 0, 1, 2, \cdots, 7)$.

Problem 5

When we define the Fourier transform $F(\omega)$ of a function $f(t)$ as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt \ , \tag{1}$$

its inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t)\, d\omega \ . \tag{2}$$

On the other hand, when we define the Fourier coefficients c_n of a function $g(t)$ which has a period of t_0 as

$$c_n = \frac{1}{t_0} \int_0^{t_0} g(t) \exp(-in\omega_0 t)\, dt \qquad (n = 0, \pm 1, \pm 2, \cdots) \ , \tag{3}$$

the Fourier series expansion of $g(t)$ is given by

$$g(t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_0 t) \ . \tag{4}$$

Here i is the imaginary unit, and $\omega_0 = \dfrac{2\pi}{t_0}$. Answer the following questions.

I. We define a function $f_d(t)$ using a function $f(t)$ and the delta function $\delta(t)$ as

$$f_d(t) = \sum_{k=-\infty}^{\infty} f(kT_0)\, \delta(t - kT_0) \qquad (-\infty < t < \infty) \ , \tag{5}$$

where T_0 is a positive real constant. Derive the Fourier transform $F_d(\omega)$ of the function $f_d(t)$ using Equation (1).

II. Using the result of Question I, show that $F_d(\omega)$ is a periodic function with respect to ω, and find the period common to any arbitrary function $f(t)$.

III. Let $f(t)$ be an arbitrary periodic function with respect to t whose period is KT_0 (K: a natural number), and $f_d(t)$ be also a periodic function with a period of KT_0. Derive the Fourier coefficients d_n of $f_d(t)$ using Equation (3). For the integration interval, consider one period given by $0 - \varepsilon \leq t \leq KT_0 - \varepsilon$, where ε is a sufficiently small positive real number. In addition, the coefficients d_n have a period with respect to n. Using K, express the period L common to any function $f(t)$.

IV. We calculate d_n when $K = 6$. For such calculation, we need only $f(0), f(T_0), \cdots$, and $f(5T_0)$, and only have to obtain $d_0, d_1, \cdots$, and d_{L-1} because of the periodicity of d_n shown in Question III. Under this circumstance, $d_n\,(n = 0, 1, \cdots, L-1)$ are elements of the L-dimensional column vector

$$\boldsymbol{d} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{L-1} \end{bmatrix} \tag{6}$$

obtained by multiplying an $L \times 6$ matrix $\boldsymbol{W}$ to the six-dimensional column vector

$$\boldsymbol{f} = \begin{bmatrix} f(0) \\ f(T_0) \\ \vdots \\ f(5T_0) \end{bmatrix}. \tag{7}$$

When a is defined as $a = \exp\left(-i\frac{2\pi}{6}\right)$, express each element of the matrix $\boldsymbol{W}$ using $a^m\,(m = 0, 1, 2)$ and T_0.

Problem 6

There are k red balls (k: an integer larger than or equal to 2), k white balls, and two bags (Bag A and Bag B). Two balls are in Bag A, and all the other balls are in Bag B. The state of Bag A is one of the three states, S_0: two red balls, S_1: one red ball and one white ball, and S_2: two white balls.

One operation is defined as a procedure where one ball is randomly chosen from Bag A and put into Bag B, and then, one ball is randomly chosen from Bag B and put back into Bag A.

Answer the following questions.

I. When the initial state of Bag A is S_1, calculate the probability of each state of Bag A after the first operation.

II. When the initial state of Bag A is S_0, calculate the probability of each state of Bag A after the second operation.

III. When the probability of each state of Bag A after the n-th operation (n: a natural number) is given, express the probability of each state of Bag A after the $(n+1)$-th operation, using the given probabilities.

IV. Obtain the probability of each state of Bag A after a sufficiently large number of operations.