2011
The Graduate School Entrance Examination
Physics
9:00 am－11:00 am

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets whether English or Japanese until the start of the examination is announced.

2. Notify if you find any page missing, out of order or unclear.

3. Answer two problems out of the four problems of the problem booklet.

4. You are given two answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2, P 3, P 4) on that sheet and also the class of master's course (M) and doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided.

Problem 1

Consider wave motion of water in a box shown in Figure 1.1. The motion is allowed only in the horizontal (x) and depth (y) directions. The motion is homogenous in the thickness direction. Assume that the velocity $u(t, x)$ of the water motion in the horizontal direction does not have a variation in the depth direction, that is, the whole water column shaded in Figure 1.1 moves horizontally with the velocity of $u(t, x)$. Assume that the height variation of the water column is induced only by the variation of the water volume in the column and that the vertical water flow caused by the volume variation in the column is negligible.

W is the width of the box, and H is the depth of the water in the stationary condition. $h(t, x)$ is the wave height measured from the stationary water level at time t and horizontal position x. Then the depth of water is given by $H + h(t,x)$. Use ρ for the density of water, and g for the gravitational acceleration.

Neglect friction between the bottom of the box and the water, friction between the water surface and the air, and surface tension of the water. Assume the water height $h(t, x)$ is negligibly small compared to H.

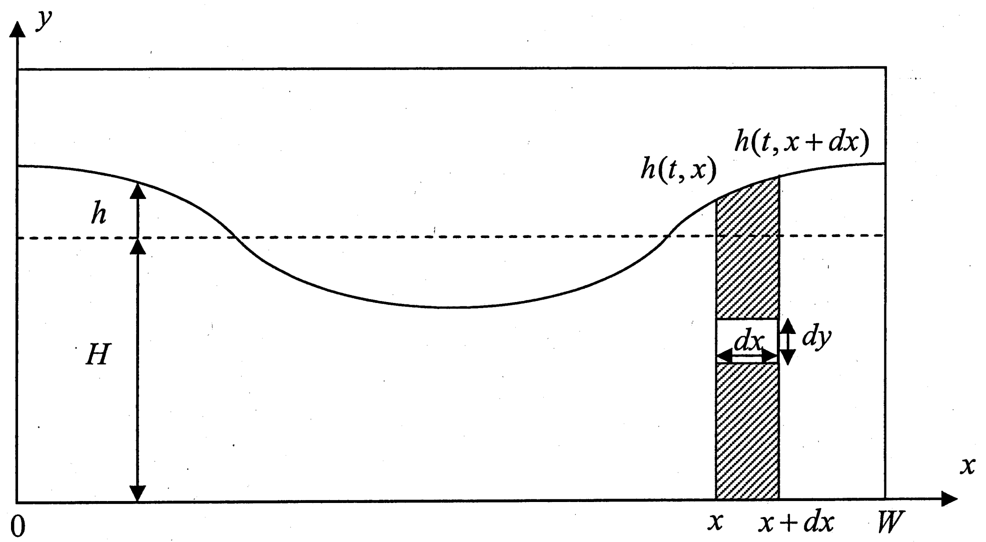

Figure 1.1

I. Express partial derivative of $u(t, x)$ with respect to time t by considering the difference in water pressure in the x direction acting on a small volume element $dxdy$ shown in Figure 1.1. Use only the symbols defined above.

II. Express partial derivative of $h(t, x)$ with respect to time t using the symbols defined above. Note that the variation of the water volume in the shaded area is equal to the difference between the volume of water flowing in and out of the shaded area in the x direction.

III. Show the partial differential equations for $u(t, x)$ and $h(t,x)$ by using the results of Questions I and II.

IV. Express the propagation velocity of the wave by using the result of Question III.

V. Obtain the longest period of the wave motion among the waves which take extreme values at the center and both ends of the box and whose phases at the center and both ends are opposite, as shown in Figure 1.1. Note that $h(t, x)$ can be written as a product of a function of t and a function of x.

Problem 2

Consider a transformer as depicted in Figure 2.1. It has a troidal magnetic core with a mean circumference l and cross section S, and it has primary and secondary windings of N_1 and N_2 turns, respectively. Here the magnetic permeability of the core is denoted as μ. Assume that there is no leakage of magnetic flux from the core, no electric or magnetic loss in the core, and no electric loss in the coils. Unless otherwise specified, there is no magnetic saturation in the core.

I. An alternating voltage of $v_1 = V_1 \cos\omega t$ is applied to the primary winding and the secondary winding left unconnected as depicted in Figure 2.2. Obtain current i_1 at the primary winding, and voltage v_2 at the secondary winding.

II. An alternating voltage of $v_1 = V_1 \cos\omega t$ is applied to the primary winding and a flat plate capacitor connected to the secondary winding as depicted in Figure 2.3. The gap, length and width of the capacitor are d, w and a, respectively. Suppose a dielectric with thickness d and width a is inserted and fixed by $w/2$ into the capacitor. In other words, the left tip of the dielectric reaches the center of the capacitor. Denote the dielectric constants for air and the dielectric as ε_a and ε_d. Here the electric field lines are parallel within the capacitor.

1. Obtain the lowest current amplitude I_m and the corresponding angular frequency ω_m, assuming the dielectric loss is negligible.
2. A periodic force pulls the dielectric into the capacitor. Express the angular frequency Ω of the periodic force using the angular frequency ω applied to the primary winding.

III. Consider the case of Question I. In reality, the amplitude of the magnetic flux density in the core cannot exceed the limit B_s due to saturation. Obtain the maximum voltage amplitude of V_1 that may be applied to the primary winding without causing saturation.

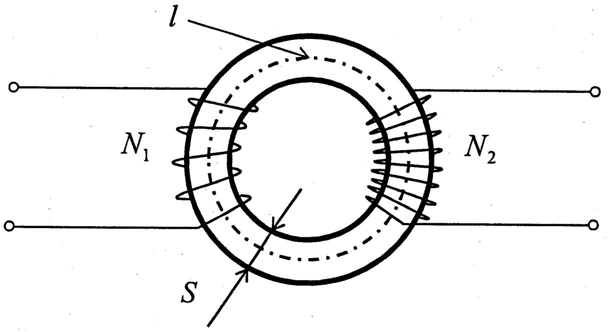

Figure 2.1

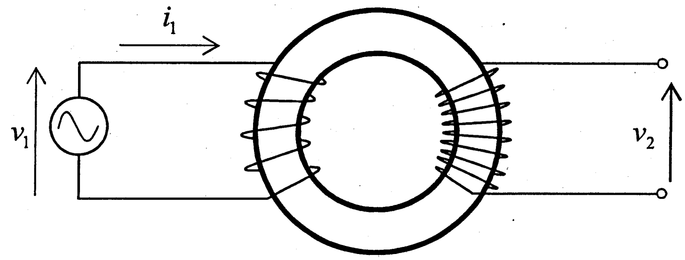

Figure 2.2

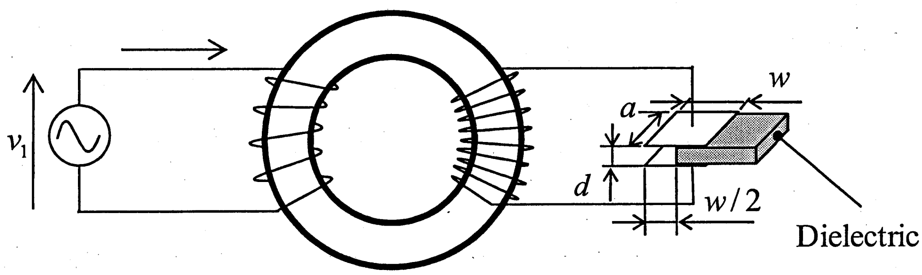

Figure 2.3

Problem 3

I. The first law of thermodynamics for a reversible system is given by $dU = TdS + d'W$ where U, T, S, and $d'W$ are internal energy, absolute temperature, entropy, and external work supplied to the system, respectively.

1. The Helmholtz free energy is given by $F = U - TS$. Show that $dF = -SdT + d'W$.
2. Explain that the change of the free energy in an isothermal process is equivalent to the external work supplied to the system.

II. Consider a reversible process in which a spring is extended. We denote the extension of the spring by x and its tension by σ. Answer the following questions, assuming that the change in the volume of the spring is negligible.

1. Derive expressions for $\left(\frac{\partial U}{\partial x}\right)_T$, $\left(\frac{\partial U}{\partial T}\right)_x$, $\left(\frac{\partial F}{\partial x}\right)_T$, and $\left(\frac{\partial F}{\partial T}\right)_x$, the first-order partial derivatives of internal energy U and free energy F with respect to x and absolute temperature T, by using x, T, σ, and entropy S.
2. Show that the following relation holds:

$$\left(\frac{\partial U}{\partial x}\right)_T = \sigma - T\left(\frac{\partial \sigma}{\partial T}\right)_x$$

3. Consider a spring whose tension σ is proportional to x when extended under isothermal condition. We denote the tension by $\sigma = k(T)x$ with a temperature dependent spring constant $k(T)$. Using $k(T)$, derive expressions for free energy F, entropy S, and internal energy U as functions of x. Use $F_0(T)$, $S_0(T)$, $U_0(T)$ for their values at $x = 0$.

III. Tension σ of a rubber string is approximately proportional to its absolute temperature T when its extension x is kept constant. Assuming that σ is proportional to x under isothermal condition, answer the following questions.

1. Setting $\sigma = AT$ $(A > 0)$, show that the internal energy of the rubber string depends only on the temperature, and that the entropy decreases as extension x is increased. Use the equation obtained in Question II, if necessary.

2. State briefly what you can speculate about the mechanism of tension of rubber strings. Use the following keywords: ideal gas, pressure, molecules, thermal motion.
3. Show that the temperature of this rubber string will increase when it is extended adiabatically. Here, let the heat capacity of the rubber string be C_x ($C_x > 0$), when the extension is kept constant.

Problem 4

A particle of mass m moves in a one-dimensional well potential given by

$$V(x) = \begin{cases} +\infty & (|x| > a/2) \\ 0 & (|x| \le a/2) \end{cases},$$

where a is a positive constant. The Hamiltonian H is given by

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x),$$

where $\hbar = h/2\pi$ is the reduced Planck's constant.

I. Solve the Schrödinger equation $H\psi(x) = E\psi(x)$, and find $\psi_n(x)$ and E_n for the stationary state of the particle confined in the potential. Here $\psi_n(x)$ is the normalized wave function and E_n the corresponding Eigen-energy, and n is a positive integer index which increases with energy. Let the lowest energy state be given by $n = 1$.

II. Consider the case where the normalized wave function of the particle at time $t = 0$ is given by

$$\Psi(x, t = 0) = A\left(4\cos^3\frac{\pi}{a}x - 5\cos\frac{\pi}{a}x\right),$$

where A is a constant.

1. Derive the value of A and express $\Psi(x, t = 0)$ as a linear combination of $\psi_n(x)$ obtained in Question I. Use the relation $4\cos^3\theta = 3\cos\theta + \cos 3\theta$, if necessary.
2. Obtain the energy expectation value $\langle E \rangle$ of this particle.
3. By solving the time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t),$$

obtain an expression for $\Psi(0,t)$, the wave function at $x = 0$ and time t.

4. Obtain the probability that the particle is observed at $x = 0$ and time t.