2014
The Graduate School Entrance Examination
Physics
9:00 am－11:00 am

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer two problems out of the four problems in the problem booklet.

4. You are given two answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2, P 3, P 4) on that sheet and also the class of the master's course (M) and doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Consider a spherical shell whose thickness is infinitesimally thin and area density (mass per unit surface area) is uniform, with mass of M, and radius of a. Neglecting the deformation of the sphere, answer the following questions to obtain the moment of inertia around its central axis.

1. Figure 1.1 shows the spherical shell around the central axis l which passes through the center O. Consider a thin circular ring on the shell which is subtended by the center angles φ and $\varphi + d\varphi$ ($d\varphi \ll 1$) formed at the center O from the axis l. Given the area density of the shell is σ, express the mass dM of the thin circular ring, in terms of a, φ, $d\varphi$, and σ.
2. Show that the moment of inertia I around its central axis l can be expressed as the form, $I = \frac{2}{3}Ma^2$.

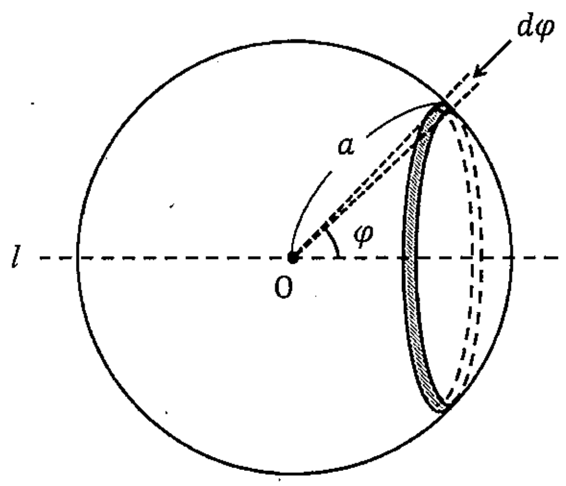

Figure 1.1

II. As shown in Figure 1.2, the x and y axes are defined in the horizontal and vertical directions, respectively. A spherical shell A, with mass of M and radius of a, is held to rotate freely around the central axis which is perpendicular to the xy plane. Another spherical shell B, with mass of m ($m<M$) and radius of b ($b<a$), is placed on the top of stationary A, and then gently made to roll to the right. Shell B starts to roll from the stationary state and eventually parts from A. Assume that when B is rolling over A, there is no slip between the two bodies. Both the centers of A and B always stay on the xy plane, and the rotation axis of B is perpendicular to the xy plane.

The angle between the vertical axis and the line connecting the centers of A and B is θ, and thereby the angular velocity and angular acceleration of the center of B relative to the center of A are expressed as $\dot{\theta}$ and $\ddot{\theta}$, respectively. Here, angular velocities of A and B around their centers are ω_A and ω_B, respectively. Their rotational directions are defined as depicted in Figure 1.2. Since there is no slip at the contact point between A and B while B is rolling over A,

$$a\omega_A = (a+b)\dot{\theta} - b\omega_B . \qquad (1)$$

The normal and friction forces at the contact point between A and B are R and F, respectively. Gravitational acceleration is g. Assuming that the thickness and deformation of the spherical shells are negligible, and that the area density of the shells is uniform, answer the following questions.

1. For both shells A and B, derive the equations of rotational motion around their respective centers of gravity.
2. With respect to the translational motion of the center of gravity of B, derive the equations of motion in the tangential direction (the x' direction in Figure 1.2) and the normal direction (the y' direction in Figure 1.2).
3. Express the angular velocities ω_A and ω_B, in terms of M, m, a, b, and $\dot{\theta}$.
4. Express the angular velocity $\dot{\theta}(>0)$ in terms of g, M, m, a, b, and θ.

5. Given that $\theta = \theta_c$ when B is parting from A, determine θ_c, in terms of M and m.

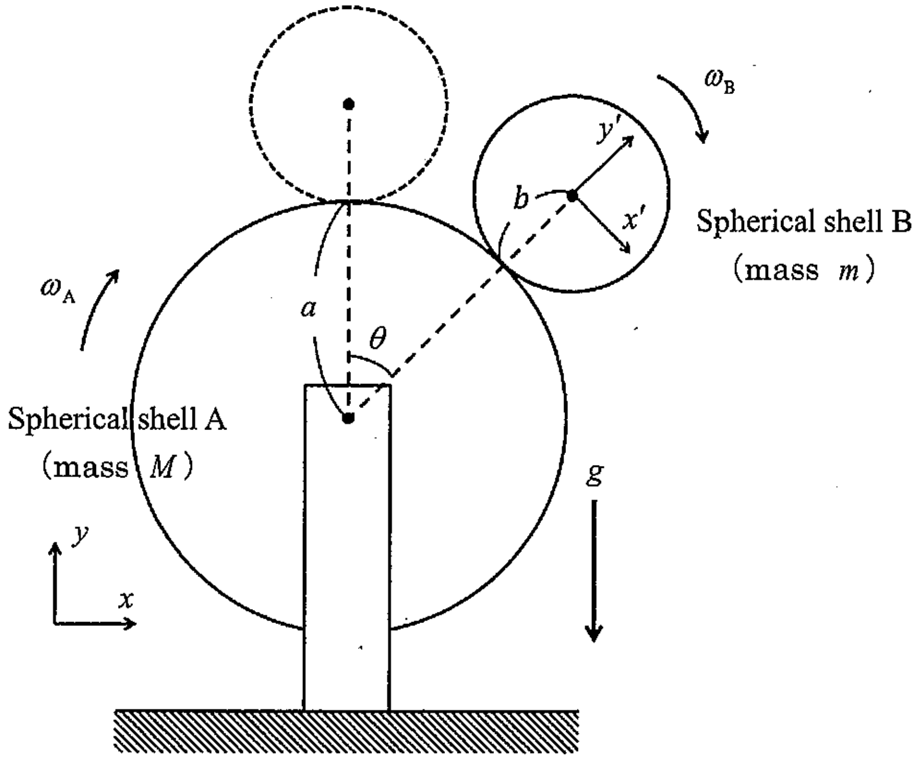

Figure 1.2

Problem 2

A flat conducting plate with an infinite area is placed in vacuum. The plate is grounded and its surface is on the *yz* plane including the origin O in the *xyz* coordinates. Either a point charge or an electric dipole is placed at $(a, 0, 0)$ $(a > 0)$. Consider an arbitrary point $\mathrm{A}(x, y, z)$ in $x > 0$. Given that the dielectric permittivity of vacuum is ε_0, answer the following questions. In all cases, describe the process with which you reached your answers.

I. As shown in Figure 2.1, a point charge of $+q$ is placed at $\mathrm{B}(a, 0, 0)$. The electric field in $x > 0$ is equivalent to the case where the plate is removed and an image charge of $-q$ is placed at $\mathrm{C}(-a, 0, 0)$. Let r_{AB} denote the distance from A to B, r_{AC} the distance from A to C, dS the small area surrounded by concentric circles around O of radius r' and $r' + dr'$, which are on the plate, and dQ the induced charge in dS.

1. Derive the electric potential V_{A} at A.
2. Derive E_x, which is the x component of the electric field at A.
3. Derive the surface charge density σ of the plate.
4. Derive dQ.
5. Calculate the total charge Q induced on the flat conducting plate.

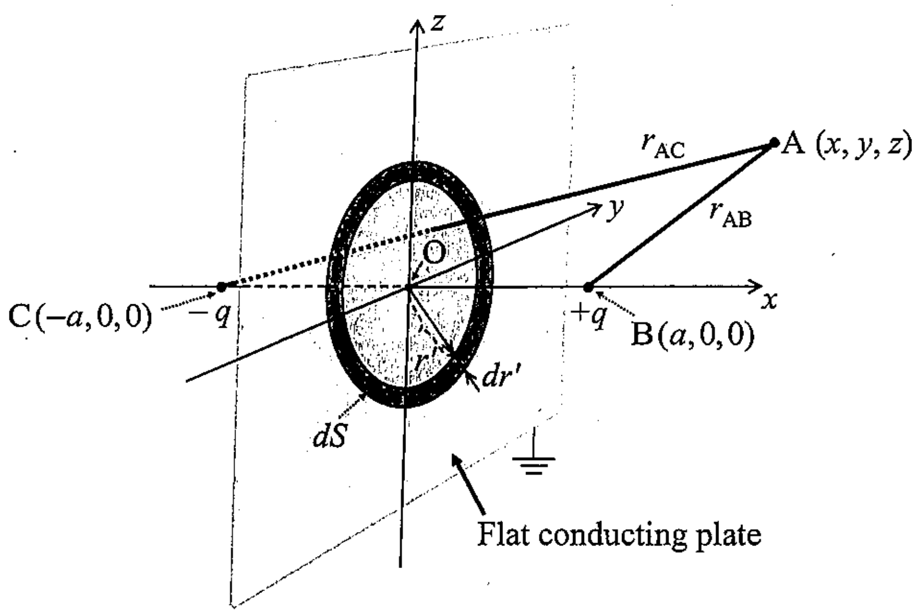

Figure 2.1

II. An electric dipole is placed such that its center is located at $(a,0,0)$. The dipole consists of charges of $+q$ and $-q$ separated by distance b, and has an electric dipole moment of $\boldsymbol{p}$ $(|\boldsymbol{p}| = qb)$. The position vector of A from the center of the dipole is denoted as $\boldsymbol{r}$ ($r=|\boldsymbol{r}|$). Consider the case where b is significantly smaller than r ($b \ll r$). When the flat conducting plate is removed, the electric potential V at A originating from $\boldsymbol{p}$ is approximated as $V=(\boldsymbol{p}\cdot\boldsymbol{r})/(4\pi\varepsilon_0 r^3)$.

Let $\boldsymbol{p}_1$ denote the moment of the electric dipole located at $(a, 0, 0)$ as shown in Figure 2.2(a), and $\boldsymbol{p}_2$ the moment of its image dipole. Note that $\boldsymbol{p}_1$ is parallel to the x axis. Also, let $\boldsymbol{r}_1$ ($r_1=|\boldsymbol{r}_1|$) denote the position vector of A from $(a, 0, 0)$, θ_1 the angle between $\boldsymbol{p}_1$ and $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ ($r_2=|\boldsymbol{r}_2|$) the position vector of A from $(-a, 0, 0)$, and θ_2 the angle between $\boldsymbol{p}_2$ and $\boldsymbol{r}_2$. In Figure 2.2(a), let $V_{//}$ denote the electric potential at A, and $\sigma_{//}$ the surface charge density of the plate.

1. Derive $V_{//}$.
2. Derive $\sigma_{//}$.

Let $\boldsymbol{p}_3$ denote the moment of the electric dipole located at $(a, 0, 0)$ as shown in Figure 2.2(b), and $\boldsymbol{p}_4$ the moment of its image dipole. Note that $\boldsymbol{p}_3$ is parallel to the z axis. Also, $\boldsymbol{r}_3$ ($r_3=|\boldsymbol{r}_3|$), $\boldsymbol{r}_4$ ($r_4=|\boldsymbol{r}_4|$), θ_3, and θ_4 are defined as shown in Figure 2.2(b). In Figure 2.2(b), let $V_{\perp}$ denote the electric potential at A, and $\sigma_{\perp}$ the surface charge density of the plate.

3. Derive $V_{\perp}$.
4. Calculate $\sigma_{\perp}/\sigma_{//}$ at $(0, 0, a)$ of the flat conducting plate.
5. Draw a sketch of $\sigma_{\perp}$ at $y=0$ as a function of z. If the z dependence of $\sigma_{\perp}$ at $y=0$ has point(s) of maximum and/or minimum, derive the value(s) and mark the point(s).

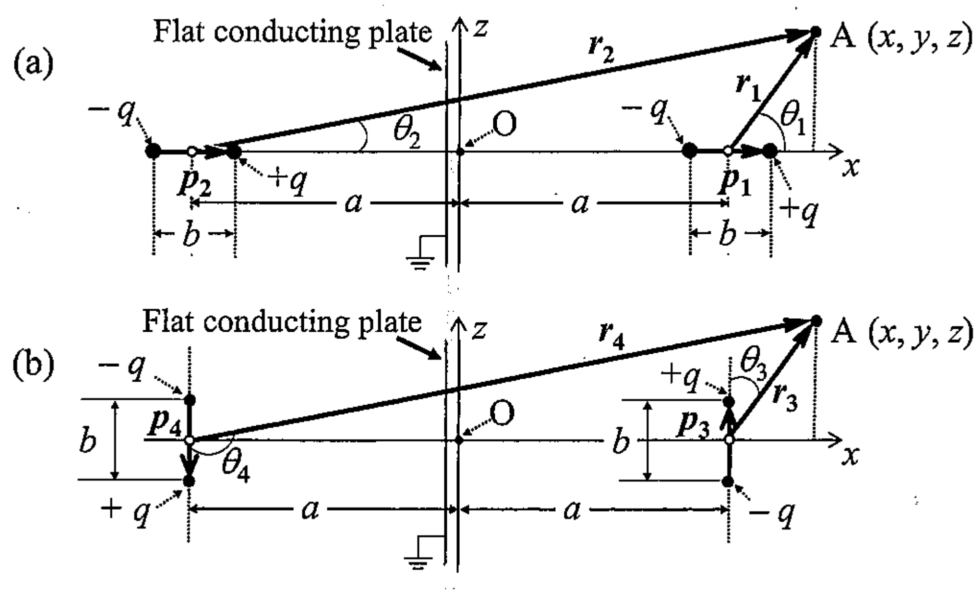

Figure 2.2

Problem 3

Consider a thermal process where gas serves as the working fluid. Here, P is the pressure of the working fluid, V is the specific volume (i.e., the volume per mole), T is the absolute temperature, R is the gas constant, U is the internal energy per mole, and S is the entropy per mole. Moreover, the enthalpy per mole is defined as $H = U + PV$, the specific heat at constant volume is $C_V = (\partial U/\partial T)_V$, the specific heat at constant pressure is $C_P = (\partial H/\partial T)_P$, and the ratio of specific heat is $\kappa = C_P/C_V$. All changes in state are assumed to be quasi-static and therefore, $dU = TdS - PdV$ holds true. Answer the following questions.

I. Show that Equations (1) ~ (4) hold true, regardless of the equation of state for the working fluid.

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V, \tag{1}$$

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_P, \tag{2}$$

$$C_P - C_V = T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_P, \tag{3}$$

$$\left(\frac{\partial P}{\partial V}\right)_S = \kappa\left(\frac{\partial P}{\partial V}\right)_T. \tag{4}$$

If necessary, use the following from the Maxwell relations,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V, \tag{5}$$

and the following chain rules,

$$\left(\frac{\partial S}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_S\left(\frac{\partial V}{\partial S}\right)_T = -1, \tag{6}$$

$$\left(\frac{\partial S}{\partial T}\right)_P\left(\frac{\partial T}{\partial P}\right)_S\left(\frac{\partial P}{\partial S}\right)_T = -1. \tag{7}$$

Ⅱ. Consider the case where the working fluid obeys the equation of state for an ideal gas,

$$PV = RT. \tag{8}$$

Here, R and C_P are assumed to be constant. Answer the following questions.

1. Show that κ is constant using Equation (3).
2. Show that the following relation holds true during an adiabatic process using Equation (4),

$$PV^{\kappa} = \text{constant}. \tag{9}$$

Ⅲ. Consider an ideal thermal cycle, O→A→B→C→D→E→O, (inter-cooling gas-turbine cycle) expressed by the T-S diagram as shown in Figure 3.1. Assume that the working fluid obeys the equation of state for an ideal gas (Equation (8)) and that R and C_P are constant. P and T of the working fluid change through one cycle as follows,

state O (P_O, T_O) to state A (P_A, T_A) via adiabatic compression,
state A (P_A, T_A) to state B (P_B, T_B) via cooling at constant pressure,
state B (P_B, T_B) to state C (P_C, T_C) via adiabatic compression,
state C (P_C, T_C) to state D (P_D, T_D) via heating at constant pressure,
state D (P_D, T_D) to state E (P_E, T_E) via adiabatic expansion, and
state E (P_E, T_E) to state O (P_O, T_O) via cooling at constant pressure,

where $P_D = P_C \geqq P_B = P_A \geqq P_O = P_E$. Moreover, $T_B = T_O$ holds true assuming an ideal heat exchange during Process A to B (inter-cooling process). The temperature ratio of maximum over minimum through the cycle is $\tau = T_D/T_O$, the overall pressure ratio is $r = P_C/P_O$, the pressure ratio during Process O to A is $r_{OA} = P_A/P_O$, and the work done by the working fluid per mole during one cycle is W. Answer the following questions.

1. Express T_C in terms of T_O, κ, r, and r_{OA}.
2. Express T_E in terms of T_O, κ, τ, and r.
3. In this cycle, W is given as

$$W = -C_P\{(T_A - T_O) + (T_C - T_B) + (T_E - T_D)\}. \tag{10}$$

Express W in terms of C_P, T_O, κ, τ, r, and r_{OA}.

4. Assuming T_O, τ, and r are fixed, express r_{OA} using r when W takes the

maximum value. Moreover, express the maximum value of W in terms of C_P, T_O, κ, τ, and r.

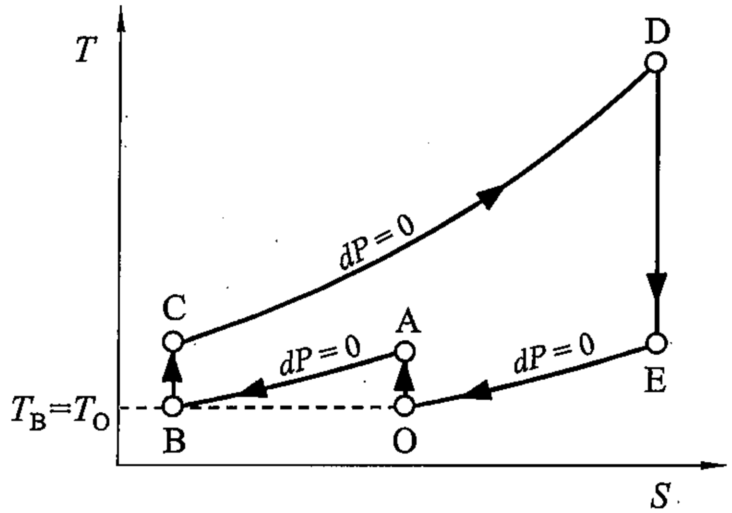

Figure 3.1

Problem 4

On the xy plane shown in Figure 4.1, consider a point light source at $A(-a, 0)$ $(a > 0)$. A thin plano-convex lens is placed with its optical axis on the x axis and the flat surface at $x = 0$. The convex surface of the lens is spherical with a radius of curvature of R. The aperture radius of the lens is r, and the thickness of the lens is $T(y_0)$ at $y = y_0$ in the xy plane. The maximum thickness of the lens is T_0. In this situation, the light emitted from A is focused to spot $B(b, 0)$ $(b > 0)$. This imaging system including the light source and the lens are placed in air. Here, the refractive indices of air and the lens are 1 and n, respectively. A light ray passing through $C(0, y_0)$ $(|y_0| < r)$ exits from the lens at D on the convex surface.

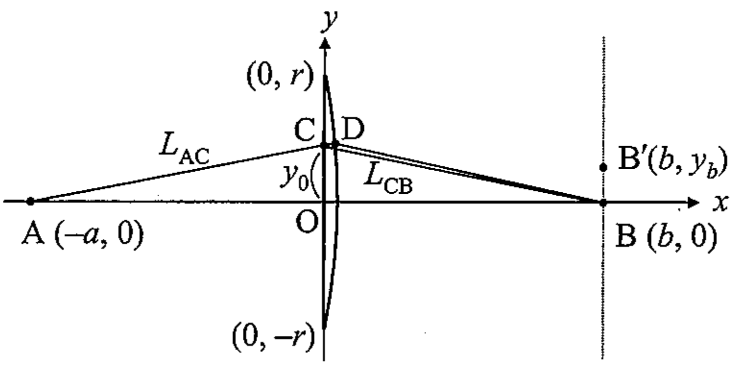

Figure 4.1

I. Among the light rays that propagate from A to B, consider the light ray passing through C and D. Given that the geometric lengths of AC and CB are L_{AC} and L_{CB}, respectively, answer the following questions.

1. Obtain expressions for L_{AC}, L_{CB}, and $T(y_0)$.
2. Derive polynomial approximations for L_{AC}, L_{CB}, and $T(y_0)$, under the conditions that $|y_0| \ll a, b, R$. Here, you may use the following approximation formula for real numbers m and s, when $|s| \ll 1$.

$$(1 + s)^m \approx 1 + ms. \tag{1}$$

3. Furthermore, if the angle subtended by the light ray and the x axis is small, the geometric lengths of CD and DB can be approximated as $T(y_0)$ and $L_{CB} - T(y_0)$, respectively. Considering the optical path length in the lens is n times the geometric length, derive the expression for the optical path length $L(y_0)$ of the light ray from A to B passing through C and D.
4. According to Fermat's principle, when A is imaged to B, $L(y_0)$ is insensitive to y_0. Using this principle, obtain the relationship between a and b. Then, derive the focal length of the lens.

II. In reality, the spot at B has a finite size. This can be explained by the wave nature of light. Although light is emitted from the point source in all directions, consider the case where only the light traveling adjacent to the xy plane is let through by placing a slit on the flat surface of the lens. In this situation, let us determine the intensity distribution on the focal plane $x = b$. Here, the optical path length between A and $B'(b, y_b)$ through C and D is given as $L'(y_0, y_b)$. The complex amplitude of electric field of light $E(y_b)$ at B' can be obtained by superposition of light waves traveling through different optical paths. Considering that the phase change due to the optical path length is $kL'(y_0, y_b)$, where $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength, $E(y_b)$ can be approximated as

$$E(y_b) = A_0 \int_{-r}^{r} \exp\left[ikL'(y_0, y_b)\right] dy_0 \, . \qquad (2)$$

Here, A_0 is a complex constant, and i is an imaginary unit. Answer the following questions.

1. Obtain the expression for $L'(y_0, y_b)$, and show that $L'(y_0, y_b)$ is expressed as a linear equation of y_0. Here, use the same approximations that were introduced in Question I, if necessary.
2. Derive the intensity distribution $|E(y_b)|^2$ at $x = b$, using Equation (2).
3. Draw a sketch of $|E(y_b)|^2$, and obtain the value(s) of y_b where the intensity takes local minimum.