2016
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer three problems out of the six problems in the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Find the general solution of the following differential equation:

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{dy}{dx} - y = 9e^{-2x}. \qquad (1)$$

Here, e denotes the base of the natural logarithm.

II. Find the value of the following integral:

$$\int_0^1 x^m (\log x)^n \, dx. \qquad (2)$$

Here, m and n are non-negative integers.

III. We define $I(m)$ as

$$I(m) \equiv \int_0^1 x^m \arccos x \, dx. \qquad (3)$$

Here, m is a non-negative integer. Use the principal values of inverse trigonometric functions.

1. Find the value of $I(0)$.
2. Find the value of $I(1)$.
3. Express $I(m)$ in terms of m and $I(m-2)$ when $m \geq 2$.
4. Find the value of $I(m)$.

Problem 2

Consider the column vectors $\mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

I. When $\mathbf{A} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$, obtain the three-dimensional column vector $\mathbf{x}$ which meets

$$\mathbf{Ax} - \mathbf{b} = \mathbf{0}. \tag{1}$$

II. Any $m \times n$ real matrix $\mathbf{B}$ is expressed using orthonormal matrices $\mathbf{U}(m \times m)$ and $\mathbf{V}(n \times n)$ as

$$\mathbf{B} = \mathbf{U\Sigma V}^T, \quad \Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & \sigma_r & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad r = \mathrm{rank}(\mathbf{B}). \tag{2}$$

$\sigma_1, \sigma_2, \cdots, \sigma_r$ are positive real numbers, and they are called singular values of $\mathbf{B}$. $\mathbf{P}^T$ means the transposed matrix of a matrix $\mathbf{P}$. Then, express $\mathbf{BB}^T$ and $\mathbf{B}^T\mathbf{B}$ using matrices $\mathbf{U}$, $\mathbf{V}$, Σ and their transposed matrices, respectively.

Let $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2)$ in the following questions.

III. Find the eigenvalues and corresponding eigenvectors for $\mathbf{BB}^T$.

IV. Find singular values of $\mathbf{B}$ and orthonormal matrices $\mathbf{U}$ and $\mathbf{V}$ used in Equation (2).

V. Find the two-dimensional column vector $\mathbf{x}$ which minimizes the norm

$$\|\mathbf{Bx} - \mathbf{b}\|^2 = (\mathbf{Bx} - \mathbf{b})^T(\mathbf{Bx} - \mathbf{b}). \tag{3}$$

Problem 3

Consider a mapping $w = f(z)$ of a domain D on the complex z plane to a domain Δ on the complex w plane. Points on the complex z and w planes correspond to complex numbers $z = x + iy$ and $w = u + iv$, respectively. Here, x, y, u and v are real numbers, and i is the imaginary unit.

I. Let $w = \sin z$.

1. Express u and v as functions of x and y, respectively.
2. Suppose the domain $D_1 = \{(x, y) \mid 0 \le x \le \frac{\pi}{2}, y \ge 0\}$ on the z plane is transformed to a domain on the w plane. Show the transformed domain on the w plane by drawing the transformed images corresponding to the three half-lines: $x = 0$, $x = \frac{\pi}{2}$ and $x = c$ at $y \ge 0$ on the z plane. Here, c is a real constant on $0 < c < \frac{\pi}{2}$.

II. If a real function $g(x, y)$ has continuous first and second partial derivatives and satisfies Laplace's equation $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$ in a domain Ω on a plane, $g(x, y)$ is said to be harmonic in Ω.

Suppose that a function $f(z) = u(x, y) + iv(x, y)$ is holomorphic in D on the z plane:

1. Show both $u(x, y)$ and $v(x, y)$ are harmonic in D on the z plane.
2. Suppose a function $h(u, v)$ is harmonic in Δ on the w plane, show a function $H(x, y) = h(u(x, y), v(x, y))$ is harmonic in D on the z plane.

III. Suppose a function $h(u, v)$ is harmonic in the domain $\Delta_1 = \{(u, v) \mid u \ge 0, v \ge 0\}$ on the w plane and satisfies the following boundary conditions:

$$h(0, v) = 0 \quad (v \ge 0), \tag{1}$$

$$h(u, 0) = 1 \quad (u \ge 1), \tag{2}$$

$$\frac{\partial h}{\partial v}(u, 0) = 0 \quad (0 \le u \le 1). \tag{3}$$

1. Let $z = \arcsin w$ and $H(x, y) = h(u, v)$. Find the boundary conditions for $H(x, y)$ corresponding to Equations (1), (2) and (3). Use the principal values of inverse trigonometric functions.
2. Find the function $H(x, y)$ which satisfies the boundary conditions obtained in Question III.1.
3. Find $h(u,0)$ on the interval $0 \leq u \leq 1$.

Problem 4

In a three-dimensional Cartesian coordinate system xyz, consider the positional relationship among three planes defined by Equations (1)–(3), and the positional relationship among the three planes and a sphere defined by Equation (4).

$$a_{11}x + a_{12}y + a_{13}z = b_1, \quad (1)$$
$$a_{21}x + a_{22}y + a_{23}z = b_2, \quad (2)$$
$$a_{31}x + a_{32}y + a_{33}z = b_3, \quad (3)$$
$$x^2 + y^2 + z^2 = 3, \quad (4)$$

where a_{ij} and b_i $(i, j = 1, 2, 3)$ are constants.

For the three planes, let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be the coefficient matrix and $\mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$ be the augmented coefficient matrix.

I. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -c \end{pmatrix}$ where c is a positive constant.

1. Find $\mathrm{rank}(\mathbf{A})$ and $\mathrm{rank}(\mathbf{B})$.
2. Among the three planes, the plane that is tangential to the sphere defined by Equation (4) at a point P (1, 1, 1) is called Plane 1. Between the other two planes, the plane with the shorter distance to P is called Plane 2. Find the distance between P and Plane 2. Then, find the volume of the part of the sphere existing between Planes 1 and 2.

II. When the three planes intersect in a line, find $\mathrm{rank}(\mathbf{A})$ and $\mathrm{rank}(\mathbf{B})$.

III. Suppose that the three planes are tangential to the sphere at three different points. Illustrate all possible positional relationships among the three planes and the sphere. In addition, for each relationship, find $\mathrm{rank}(\mathbf{A})$ and $\mathrm{rank}(\mathbf{B})$.

Problem 5

I. A function $f(x)$ is continuous and defined on the interval $0 \le x \le \pi$. If $f(x)$ is extended to the interval $-\pi \le x \le \pi$ as an odd function, it can be expanded in the following Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin nx\right), \tag{1}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, 3, \cdots). \tag{2}$$

Here, $f(0) = f(\pi) = 0$.

1. Find the Fourier sine series for the following function $f(x)$:

$$f(x) = x(\pi - x) \quad (0 \le x \le \pi). \tag{3}$$

2. Derive the following equation using the result obtained in Question I.1,

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}. \tag{4}$$

II. A two-variable function $f(x, y)$ is continuous and defined in the region $0 \le x \le \pi$ and $0 \le y \le \pi$. Using a similar method to Question I, $f(x, y)$ can be expanded in the following double Fourier sine series:

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(B_{mn} \sin mx \sin ny\right), \tag{5}$$

$$B_{mn} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x, y) \sin mx \sin ny \, dx dy \quad (m, n = 1, 2, 3, \cdots). \tag{6}$$

Here, $f(0, y) = f(\pi, y) = f(x, 0) = f(x, \pi) = 0$.

1. Find the double Fourier sine series for the following function $f(x, y)$:

$$f(x, y) = x(\pi - x) \sin y \quad (0 \le x \le \pi, 0 \le y \le \pi). \tag{7}$$

2. Function $u(x, y, t)$ is defined in the region $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$. Obtain the solution for the following partial differential equation of $u(x, y, t)$ by the method of separation of variables:

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{8}$$

where c is a positive constant and the following boundary and initial conditions apply:

$$u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0, \tag{9}$$

$$u(x, y, 0) = x(\pi - x) \sin y. \tag{10}$$

Problem 6

Company A owns multiple factories $i \ (i = 1, 2, \cdots)$. Suppose that the probability of producing defective goods in a factory i is P_i, and that N_i goods are randomly sampled and shipped from the factory. Here, P_i is sufficiently small, and each factory does not affect any other.

I. Show the probability $f(i,k)$, which is the probability of k defective goods existing within N_i goods shipped from a factory i. Here, k is a non-negative integer.

II. Show that $f(i,k) \to \dfrac{e^{-\lambda_i} \lambda_i^k}{k!}$ when $N_i \to \infty$. Here, when calculating the limit of $f(i,k)$, λ_i is a constant, where $\lambda_i = N_i P_i$.

In the following questions, assume that $f(i,k) = \dfrac{e^{-\lambda_i} \lambda_i^k}{k!}$.

III. Suppose that goods are shipped from two factories as shown in Table 1. Find the probability of two defective goods being contained within all shipped goods.

Table 1

Factory number (i)	Probability of defectiveness (P_i)	Number of shipped goods (N_i)
1	0.01	500
2	0.02	300

IV. Find the probability of k defective goods being contained within all shipped goods under the same conditions as in Question III.

V. Suppose that $P_i = 0.001\,i$ in five factories $i \ (i = 1, 2, 3, 4, 5)$ and the same number (N_c) of goods are shipped from all these factories.

Find the maximum value of N_c for which the expected number of defective goods out of all shipped goods is equal to or less than 3.